

Factorization Method and Hydrogen-Like Atom in Constant Electromagnetic Field

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In this paper, we use the factorization method and the shape invariance condition of associated Laguerre differential equation with respect to two parameters m and n . We obtained factorized Schrodinger equations for the relativistic hydrogen-like atom in a constant electromagnetic field in terms of first order equations. These first order equations are a form of raising and lowering operators. These operators satisfy certain commutation relations forming algebra. Furthermore, we calculate the energy spectrum and bound states for constant electromagnetic field of the relativistic hydrogen-like atom.

1. Introduction

The study of two-dimensional atom in the presence of homogenous magnetic fields has been a subject of active research during the last years. Much work, computing the energy spectrum for different magnetic field strengths, has been done in the frame work of non-relativistic quantum mechanics. This problem is of practical interest in discussing single- and multiple - quantum well (super lattice) systems in semiconductor physics [1,2,3]. Here, we discuss the two dimensional Schrodinger equation with Coulomb potential $-z/r$ and a constant magnetic B field that is perpendicular to the plane. So, the corresponding Hamiltonian can be written as

$$H = \frac{1}{2}(-i\nabla + \frac{1}{2}B \times r) - \frac{z}{r} \quad (1)$$

in polar coordinates (r, θ) . The general form of the second order equation (Eqn. 1), the relativistic Hamiltonian with constant electromagnetic field, cannot be solved in closed form in terms of special functions [4]. Therefore, we factorize the Hamiltonian of quantum states in terms of a multiplication of the first-order differential operators as shape invariant equations. In this approach, the Hamiltonian is decomposed once in successive multiplication of lowering and the raising operators in such a way that the corresponding quantum states of successive levels are its eigen states. It is known that the factorization method was suggested by Darboux [5] initially and the application of this method was provided by Schrödinger in the framework of

quantum mechanics later [6]. Recently, the concept of shape invariance has been extended to ordinary differential equations and on this basis, a second - order differential operator decomposes the multiplication of ladder operators [7-15].

This paper is organized as follows. In Sec. 2, we introduce the second order differential equation corresponding to the Hamiltonian of relativistic hydrogen-like atom with constant electromagnetic field. In order to solve this equation, we need some mathematical foundation. In Sec. 3, we introduce therefore the associated Laguerre differential equation. As mentioned in Refs. [16-20], we can factorize the associated Laguerre differential equation with respect to parameters n and m . In Sec. 4 and Sec. 5, we consider the magnetic and the electric case of hydrogen-like atom, respectively. Moreover, by using associated Laguerre equation, we factorize the corresponding second order equations in terms of first order equations. This led us to obtain the energy spectrum and raising and lowering operators with respect to n and m . Finally, we discuss the relativistic, two dimensional hydrogen-like atom in a constant electromagnetic field in Sec. 6. However, in this case, we cannot factorize the second order equation in terms of the first order equations. Therefore, we simply obtain the energy spectrum. It is because of the absence of the shape invariance condition.

2. Hamiltonian of relativistic two-dimensional hydrogen-like atom in a constant magnetic field

The covariant generalization of the Klein-Gordon equation in the presence of electromagnetic interactions takes the form [1, 5]

$$(g^{\mu\nu}(\Delta_\mu - \frac{i}{c}A_\mu)(\Delta_\nu - \frac{i}{c}A_\nu) - c^2)\Psi = 0 \quad (2)$$

Where, $g^{\mu\nu}$ is the contravariant metric tensor and Δ_μ is the covariant derivative. We work in a 2+1 space-time and the metric tensor $g^{\mu\nu}$ in polar coordinates (t, r, θ) takes the form

$$g_{\mu\nu} = \text{diag}(-1, 1, r^2) \quad (3)$$

The vector potential A^μ associated with the Coulomb and the constant magnetic field interaction is

$$A^\mu = (-\frac{z}{r}, 0, -\frac{Br^2}{2}) \quad (4)$$

Using Eqn. 4, we can verify that the magnetic and electric fields satisfy the following invariant relations

$$F_{\mu\nu}F^{\mu\nu} = 2(B^2 - E^2) = 2(B^2 - \frac{z^2}{r^4}), \quad (5)$$

$$E \cdot B = 0 \quad (6)$$

Where, $F^{\mu\nu}$ is the electromagnetic field strength tensor having three independent components in (2+1) space-time. In fact, Eqns. 4 and 5 tell us that A^μ is associated with a 2D Coulomb atom in a constant magnetic field perpendicular to the plane where the particle is located. The corresponding E and B can be written in polar coordinates as

$$E = -\frac{z}{r^2}e_r \quad (7)$$

$$B = Be_z \quad (8)$$

As we know, the vector potential do not depend on time and the angular variable θ , so the solution of the Klein-Gordon equation (Eqn. 2) can be written as

$$\Psi(r, \theta, t) = R(r)r^{-\frac{1}{2}}e^{(ipt - Et)} \quad (9)$$

The function $R(r)$ given above satisfies the second order ordinary differential equation

$$\frac{d^2R(r)}{dr^2} + (\frac{\gamma}{r^2} + \eta + \theta r^2 + \frac{\psi}{r})R(r) = 0 \quad (10)$$

Where,

$$\begin{aligned} \gamma &= \frac{1}{4} - p^2 + \frac{Z^2}{c^2}, & \eta &= \frac{E^2}{c^2} - c^2 + \frac{pE}{c}, \\ \theta &= -\frac{1}{4} \frac{B^2}{c^2}, & \psi &= \frac{2EZ}{c^2} \end{aligned} \quad (11)$$

We are looking for the solution of Eqn. 11 and consider three cases. In the first case, we take into account the magnetic field, whereas in the second case, we write the Coulomb field. We show that the energy, and the raising and lowering operators for two cases are different, but they have the same shape invariance condition. We note here that in the third case, a system with electromagnetic potential, we cannot factorize the corresponding equation. Thus it does not satisfy the shape invariance condition.

3. Mathematical foundations

With the factorization approach, we compute the energy spectrum \mathcal{E} and also bound states $\Psi(\rho, \phi)$ through comparison of the differential equation given in Eqn. 10 with the associated Laguerre differential equation in an appropriate manner. We also factorize the second order differential equations into new sets of operators $A^- A^+$ and in shape invariant form, which are the first order differential equations. This process is called the factorization method. To start, we need to recall that for the real parameters $\alpha > -1$ and $\beta > 0$, the associated Laguerre differential equation corresponding to $L_{n,m}^{(\alpha,\beta)}(z)$ in the interval $z \in (0, \infty)$ is introduced [16 -18] as

$$zL_{n,m}^{\prime(\alpha,\beta)}(z) + [1 + \alpha - \beta z]L_{n,m}^{\prime(\alpha,\beta)}(z) + [(n - \frac{m}{2})\beta - \frac{m}{2z}(\alpha + \frac{m}{2})]L_{n,m}^{(\alpha,\beta)}(z) = 0 \quad (12)$$

Here, the indices n and m are non-negative integers for $0 \leq m < n$. The associated Laguerre function, $L_{n,m}^{(\alpha,\beta)}(z)$, as the solution of the differential equation (Eqn. 12), which has the following Rodriguez representation

$$L_{n,m}^{(\alpha,\beta)}(z) = \frac{a_{n,m}(\alpha,\beta)}{z^{\frac{\alpha+m}{2}} e^{-\beta z}} \left(\frac{d}{dx}\right)^{n-m} (z^{\alpha+n} e^{-\beta z}) \quad (13)$$

Where, $a_{n,m}(z)$, is the normalization coefficient. As mentioned in Refs. [16-18] and [19- 20], we can write the associated Laguerre differential equation (Eqn. 12) as the following shape invariant equations with respect to the parameter m

$$A_m^+(z) A_m^- L_{n,m}^{(\alpha,\beta)}(z) = (n-m+1) \beta L_{n,m}^{(\alpha,\beta)}(z), \quad (14)$$

$$A_m^-(z) A_m^+ L_{n,m-1}^{(\alpha,\beta)}(z) = (n-m+1) \beta L_{n,m-1}^{(\alpha,\beta)}(z)$$

$$A_{n,m}^+(z) A_{n,m}^- L_{n,m}^{(\alpha,\beta)}(z) = (n-m)(n+\alpha) \beta L_{n,m}^{(\alpha,\beta)}(z), \quad (17)$$

$$A_{n,m}^-(z) A_{n,m}^+ L_{n-1,m}^{(\alpha,\beta)}(z) = (n-m)(n+\alpha) \beta L_{n-1,m}^{(\alpha,\beta)}(z)$$

Where the differential operators are functions of the parameters n and m , which are obtained, respectively, as

$$A_{n,m}^+(z) = z \frac{d}{dz} - \beta z + \frac{1}{2}(2n+2\alpha-m), \quad (18)$$

$$A_{n,m}^-(z) = -z \frac{d}{dz} + \frac{1}{2}(2n-m)$$

Note that the shape invariant equation (Eqn. 14) can be written as the raising and lowering relations

$$A_{n,m}^+(z) L_{n-1,m}^{(\alpha,\beta)}(z) = \sqrt{(n-m)(n+\alpha)} L_{n,m}^{(\alpha,\beta)}(z), \quad (19)$$

$$A_{n,m}^-(z) L_{n,m}^{(\alpha,\beta)}(z) = \sqrt{(n-m)(n+\alpha)} L_{n-1,m}^{(\alpha,\beta)}(z)$$

The above mentioned method, after some calculations, leads to the following normalization coefficient

$$a_{n,m}(\alpha,\beta) = (-1)^m \sqrt{\frac{\beta^{\alpha+m+1}}{\Gamma(n-m+1)\Gamma(n+\alpha+1)}} \quad (20)$$

Where, the explicit forms of operators $A_m^+(z)$ and $A_m^-(z)$ are given below as

$$A_m^+(z) = \sqrt{z} \frac{d}{dz} - \frac{m-1}{2\sqrt{z}}, \quad (15)$$

$$A_m^-(z) = -\sqrt{z} \frac{d}{dz} - \frac{2\alpha+m-2\beta z}{2\sqrt{z}}$$

One may write down the shape invariance equation (Eqn. 14) as the raising and lowering relations

$$A_m^+(z) L_{n,m-1}^{(\alpha,\beta)}(z) = \sqrt{(n-m+1)\beta} L_{n,m}^{(\alpha,\beta)}(z), \quad (16)$$

$$A_m^-(z) L_{n,m}^{(\alpha,\beta)}(z) = \sqrt{(n-m+1)\beta} L_{n,m-1}^{(\alpha,\beta)}(z)$$

On the other hand, the associated Laguerre differential equation (Eqn. 12) can be factorized with respect to parameter n , for a given m , as [16,17]

Also, the normalization coefficient (Eqn. 20) has been so chosen that the associated Laguerre functions $L_{n,m}^{(\alpha,\beta)}(z)$, with the same m but with different n 's with respect to the inner product with the weight functions $z^\alpha e^{-\beta z}$, form an ortho-normal set in the interval $0 \leq z < \infty$ and

$$\int_0^\infty L_{n,m}^{(\alpha,\beta)}(z) L_{n',m}^{(\alpha,\beta)}(z) z^\alpha e^{-\beta z} dz = \delta_{n,n'} \quad (21)$$

4. Magnetic case of hydrogen atom

Here, we shall consider the magnetic case of Eqn. 10 and therefore, we set $E = 0$ in Eqn. 11. Thus, we have

$$\frac{d^2 R(r)}{dr^2} + \left(\frac{\gamma}{r^2} + \eta' + \theta r^2 + \frac{\psi}{r} \right) R(r) = 0 \quad (22)$$

Where, $\eta' = c^2 - \frac{pB}{c}$.

By using the following change of variable,

$$R(r) = r^D e^{-\frac{Ar^2}{2}} U(r) \tag{23}$$

Where, $A = \frac{B}{2c}$ and $D = \frac{1}{2} + \sqrt{p^2 - \frac{z^2}{c^2}}$.

Then the above equation will be as

$$U''(r) + [\frac{2D}{r} - 2Ar]U'(r) + [A^2r^2 - 2DA + \frac{D(D-1)}{r^2} - A + \frac{\gamma}{r^2} + \eta' + \theta r^2]U(r) = 0 \tag{24}$$

After some calculation, we have

$$U''(r) + [\frac{2D}{r} - 2Ar]U'(r) + [\eta' - 2DA - A]U(r) = 0 \tag{25}$$

Also with the following change of variable, we get

$$x = Ar^2, U(r) = g(x) \tag{26}$$

Therefore, the above equation will be

$$xg''(x) + [D + \frac{1}{2} - x]g'(x) + [\frac{\eta}{4A} - \frac{D}{2} - \frac{1}{4}]g(x) = 0 \tag{27}$$

If we compare this equation with Laguerre equation, we find $\beta = 1$, $\alpha \geq -1$, and $m = 0$.

Comparing two equations, we get $g(x) = L_{n,0}^{(\alpha,1)}(z)$.

Thus the energy spectrum is given by

$$\epsilon = c^2 \sqrt{1 + \frac{1}{c^2}(n + \frac{pB}{c^3} + \frac{B^2}{4c^2}(2\alpha + 1))} \tag{28}$$

Where, $\alpha = D - \frac{1}{2} = \sqrt{p^2 - \frac{z^2}{c^2}}$ and

$$n = \frac{\eta'}{4A} - \frac{D}{2} - \frac{1}{4}.$$

We note here that the non-linear equation (Eqn. 22) realizes the unitary irreducible representation of SU(1, 1) group. The above eigenvalues of such representations are restricted from below and equal to

$$\epsilon = c^2 \sqrt{\frac{n^2}{c^2} + 1 + K} \tag{29}$$

Where, $K = \frac{1}{c^2}(\frac{pB}{c^3} + \frac{B^2}{4c^2}(2\alpha + 1))$.

Now, we are going to factorized the second order equation to a first order equation with respect to Eqn. 18 as

$$A_n^+ = x \frac{d}{dx} - x + n + \alpha, \tag{30}$$

$$A_n^- = -x \frac{d}{dx} + n$$

These equations lead us to expressions

$$A_n^+ L_{n-1}^{(\alpha,1)}(x) = \sqrt{n(n+1)} L_n^{(\alpha,1)}(x), \tag{31}$$

$$A_n^- L_n^{(\alpha,1)}(x) = \sqrt{n(n+1)} L_{n-1}^{(\alpha,1)}(x)$$

Therefore, we can write the equation as

$$A_n^+(x) A_n^- L_n^{(\alpha,1)}(x) = n(n+1) \beta L_n^{(\alpha,1)}(x), \tag{32}$$

$$A_n^-(x) A_n^+ L_{n-1}^{(\alpha,1)}(x) = n(n+1) \beta L_{n-1}^{(\alpha,1)}(x)$$

Using Eqn. 23, one can obtain $\psi_n(r)$ as

$$\psi_n(r) = e^{-\frac{2B}{4c}r^2} r^{\frac{1}{2} + \alpha} L_n^{(\alpha,1)}(\frac{Br^2}{2c}) \tag{33}$$

and

$$\psi_n(r) = \sqrt{\frac{\Gamma(\alpha+1)}{\Gamma(n+1)\Gamma(n+\alpha+1)}} A_n^+ A_{n-1}^+ \dots A_1^+ \psi_0(r) \tag{34}$$

Where, $\psi_0(r)$ is the lowest state. Next, we want to show the shape invariance condition corresponding to the potential. In order to do that, we have the following expression

$$A_n^\pm = \pm \frac{d}{dr} + W_n(r), \tag{35}$$

$$W_n(r) = \frac{1}{4}r - \frac{D}{2}$$

Moreover, we have

$$V_n^\pm(r) = W_n^2(r) \pm \frac{d}{dr} W_n(r) \tag{36}$$

Which satisfies the equation

$$V_n^+(r) - V_{n+1}^+(r) = const. \tag{37}$$

This is called shape invariance condition. The corresponding equation taking into account the

$$rR''(r) + [1 + \alpha - \beta r]R'(r) + [\psi - \frac{B}{2}(\alpha+1) - \frac{D^2 - 2D + 2 - \alpha^2}{4r}]R(r) = 0 \tag{41}$$

So, we have

$$\begin{aligned} \frac{\beta^2}{4} &= -\eta'', \\ \psi - \frac{B}{2}(\alpha+1) &= (n - \frac{m}{2})\beta, \\ D^2 - 2D + 2 - \alpha^2 &= \frac{m}{2}(\alpha + \frac{m}{2}) \end{aligned} \tag{42}$$

The corresponding energy and bound states will then be

$$\begin{aligned} \epsilon_{n,m} &= \frac{Bc^2}{2Z} (\frac{\alpha}{2} + \frac{1}{2} + n - \frac{m}{2}), \\ U(r) &= r^{\frac{2+\alpha+U}{2}} e^{-\beta r} L_{n,m}^{(\alpha,\beta)} \end{aligned} \tag{43}$$

magnetic field, in case of hydrogen atom, has a shape invariance relation.

5. Coulomb case of hydrogen atom

Again, we return to Coulomb case of Eqn. 10 and set $B = 0$ in relations given in Eqn.11 to obtain

$$\frac{d^2U(r)}{dr^2} + (\frac{\gamma}{r^2} + \eta'' + \frac{\psi}{r})U(r) = 0 \tag{38}$$

Where, $\eta'' = \frac{B^2}{c^2} - c^2$.

Taking the following variable

$$U(r) = r^D g(r) \tag{39}$$

One, can obtain,

$$\gamma = -D(D-1) \tag{40}$$

In the next step, we multiply r to equation and choose $g(r) = R(r)V(r)$ and a comparison with Laguerre equation gives us the equation

In that case, the shape invariance of the associated Laguerre differential equation with respect to m for a given n is realized as

$$\begin{aligned} A_m^+(r)A_m^-(r)L_{n,m}^{(\alpha,\beta)}(r) &= (n-m+1)\beta L_{n,m}^{(\alpha,\beta)}(r), \\ A_m^-(r)A_m^+(r)L_{n,m-1}^{(\alpha,\beta)}(r) &= (n-m+1)\beta L_{n,m-1}^{(\alpha,\beta)}(r), \end{aligned} \tag{44}$$

Where, the differential explicit forms of the operators A_m^+ and A_m^- are given as

$$\begin{aligned} A_m^+(r) &= \sqrt{r} \frac{d}{dr} - \frac{m-1}{2\sqrt{r}}, \\ A_m^-(r) &= -\sqrt{r} \frac{d}{dr} - \frac{\alpha+D+m-2\beta r}{2\sqrt{r}} \end{aligned} \tag{45}$$

One may write down the shape invariance (Eqn.14) as the raising and lowering relations

$$A_m^+(r)L_{n,m-1}^{(\alpha,\beta)}(r) = \sqrt{(n-m+1)\beta}L_{n,m}^{(\alpha,\beta)}(r),$$

$$A_m^-(r)L_{n,m}^{(\alpha,\beta)}(r) = \sqrt{(n-m+1)\beta}L_{n,m-1}^{(\alpha,\beta)}(r) \quad (46)$$

$$A_{n,m}^+(r)A_{n,m}^-L_{n,m}^{(\alpha,\beta)}(r) = (n-m)\left(n + \frac{\alpha + D}{2}\right)L_{n,m}^{(\alpha,\beta)}(r),$$

$$A_{n,m}^-(r)A_{n,m}^+L_{n-1,m}^{(\alpha,\beta)}(r) = (n-m)\left(n + \frac{\alpha + D}{2}\right)L_{n-1,m}^{(\alpha,\beta)}(r) \quad (47)$$

Where, the differential operators as functions of the parameters n and m are calculated, respectively, as follows:

$$A_{n,m}^+(r) = r \frac{d}{dr} - \beta r + \frac{1}{2}(2n + \alpha + D - m),$$

$$A_{n,m}^-(r) = -r \frac{d}{dr} - \beta r + \frac{1}{2}(2n - m) \quad (48)$$

According to the previous procedure, the explicit forms of the raising and lowering operators

$$A_n = \pm \frac{d}{dr} + W_m(r) \quad (49)$$

Where, $W_m(r)$ is well known as the super-potential given as

$$W_m(r) = \frac{1}{4}\beta r - \frac{\alpha + D + 2m - 1}{2r} \quad (50)$$

$$V_m^\pm(r) = W_m^2(r) \pm \frac{d}{dr}W_m(r) =$$

$$\frac{\beta^2}{16}r^2 + \frac{(\alpha + D + 2m - 1)\left(\frac{\alpha + D}{2} + m - \frac{1}{2} \pm 1\right)}{2r^2} - \frac{\beta}{4}(\alpha + D + 2m - 1 \mp 1) \quad (55)$$

Which, satisfy the following shape invariance condition on the parameter m

$$V_m^+(r) = V_{m+1}^+(r) + \beta \quad (56)$$

Also, with the help of the associated Laguerre functions, $L_{n,m}^{(\alpha,\beta)}(r)$, we obtain the quantum states corresponding to the anti-symmetric partner [21-26] as follows:

$$A_m^+(r)A_m^-(r)\psi_{n,m}^{(\alpha,\beta)}(r) = (n-m+1)\beta\psi_{n,m}^{(\alpha,\beta)}(r),$$

$$A_m^-(r)A_m^+\psi_{n,m-1}^{(\alpha,\beta)}(r) = (n-m+1)\beta\psi_{n,m-1}^{(\alpha,\beta)}(r) \quad (51)$$

So

$$\psi_{n,m-1}^{(\alpha,\beta)}(r) = r^{\frac{2+\alpha+U}{2}} e^{-\beta r^2} L_{n,m}^{(\alpha,\beta)}(r) \quad (52)$$

So, from Eqn. 16, we can obtain the raising and lowering relations for the quantum states as follows:

$$A_m^+(r)\psi_{n,m-1}^{(\alpha,\beta)}(r) = \sqrt{(n-m+1)\beta}\psi_{n,m}^{(\alpha,\beta)}(r),$$

$$A_m^-(r)\psi_{n,m}^{(\alpha,\beta)}(r) = \sqrt{(n-m+1)\beta}\psi_{n,m-1}^{(\alpha,\beta)}(r) \quad (53)$$

Now, by using the Eqn. 21 and Eqn. 52, it is easily shown that the set of quantum states $\psi_{n,m}^{(\alpha,\beta)}(r)$ for a given value of m forms an ortho-normal set

$$\int_0^\infty \psi_{n,m}(r)\psi'_{n,m}(r) = \delta_{n,n'} \quad (54)$$

Where, operators A_m^+ and A_m^- are Hermitian conjugate of each other with respect to the inner product (Eqn. 54). Shape invariance equations (Eqns. 47 and 51) describe super-symmetric partner Hamiltonian with the following partner potential:

We note here that the energy spectrum depends on n and m , which is different from the magnetic case, because in that case the energy just depends on n .

6. Electromagnetic case of hydrogen atom

In the third case too, we cannot decompose the equations in terms of raising and lowering operators and for this reason, we consider the solution of this equation as

$$U(r) = r^{\frac{1}{2} + \sqrt{m^2 - \frac{z^2}{c^2}}} e^{-\frac{\beta r^2}{4c}} \sum_0^{\infty} a_n r^n \quad (57)$$

As we know, in two previous cases, the Laguerre polynomial played an important role, but we have just general form of polynomial in here, which is not the known special function. However, this polynomial does not guarantee the shape invariance condition. We note that, by recurrence relation between different values of a and by constraining n under condition $a_n = a_{n+1} = 0$ for any positive integer value of n , we can obtain the energy spectrum as

$$\mathcal{E} = c^2 \sqrt{1 + \frac{B}{c} \left(p + n + \sqrt{p^2 - \frac{z^2}{c^2}} \right)} \quad (58)$$

One can see that the energy spectrum depends only on n .

7. Conclusion

Using the factorization method, we factorized the corresponding Schrodinger equation in terms of first order operators. These first order operators lead us to obtain the bound states and energy spectrum. First, we have considered system with magnetic field and obtained energy spectrum, raising, lowering operators, and shape invariance condition (Eqn. 37). Second, we have considered system with electric field and turned off magnetic field. In this case, we calculated energy spectrum, raising, lowering, and shape invariance condition (Eqn. 56). Finally, in the third case, we discussed system in the presence of electromagnetic field. In that case, we obtained the bound states (Eqn. 57) and energy spectrum. But, neither were we able to obtain raising and lowering operators of the corresponding system nor the shape invariance condition as given in Eqn. 37 and Eqn. 56. It may be interesting to obtain some representation for raising and lowering operators. We leave this important issue for future research.

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