A New High Energy Approximation of Intrabeam Scattering for Flat Electron and Positron Beams

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We derive completely integrated formulae for emittance growth rates due to intrabeam scattering for flat electron and positron beams in the high energy limit, including the effect of lattice parameters that vary around the accelerator ring. Using the accelerator lattice for the Accelerator Test Facility's electron damping ring at KEK, we compare our results to the more complete theory as well as to other high energy approximations.

1. Introduction

With the intensities and small beam sizes that are demanded of the modern generation of charged particle accelerators, the Coulomb scattering of particles within each bunch places an ultimate limitation on the performance of those accelerators. In proton and heavy ion colliders, Coulomb intrabeam scattering (IBS) causes the beam sizes to grow, thereby limiting the luminosity lifetimes. In electron and positron beams, IBS fixes the final equilibrium beam sizes in conjunction with synchrotron radiation damping and quantum excitations. Thus, as IBS increases the equilibrium beam sizes in an electron-positron collider, the subsequent luminosity decreases. In a synchrotron light source, a small beam size reduces the line width of insertion device radiation and translates into brighter and higher spectral quality X-ray beams; thus, the growth in equilibrium beam size due to intrabeam scattering degrades the brightness and spectral quality of such beams [1]. Hence, intrabeam scattering has become an important constraint on the performance of a wide variety of modern accelerators.

Using classical methods, Piwinski derived the original theory of IBS [2] and it is summarized in Refs. [3] and [4]. He derived the theory for weak-focussing accelerators, wherein the lattice parameters that define the magnet layout are not strongly varying around the accelerator. To account for such variation, a subsequent strong-focussing theory was derived using quantum field theory techniques in Ref. [5]. At least at high accelerator beam energies, the two theories give good agreement. Thus, a high energy approximation to the original Piwinski theory was used quite successfully in Refs. [6] and [7] to analyze the effect of IBS on the luminosity lifetime of Fermilab's first major Tevatron upgrade.

Martini incorporated varying lattice parameters into the Piwinski formalism [8]. Others have done so for the Piwinski theory in certain high energy approximations, such as discussed in Refs. [9]-[14]. Bane even mathematically proved the equivalence of a modified Piwinski and Ref. [5] theories in the high energy regime.

Regardless of whether one uses a modified Piwinski without approximation or the Ref. [5] theory, it is computationally intensive. Since the formulas involving the lattice parameters must be evaluated at numerous points around the accelerator, with each point involving integrations and/or matrix manipulations, the codes run for long times on the computer. This is a severe handicap when one wants to optimize the design of an accelerator and receive quick feedback on the IBS properties of a particular design. Since most of the interest has been for accelerator beams at extremely high energies compared to the particles' rest masses, finding high energy approximations has been a big industry for many years. In this paper we derive a new high energy approximation for flat (horizontal

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phase space, called emittance, being much larger than the vertical) electron and positron beams, which is the usual equilibrium shape for electron and positron beams undergoing bending only in the horizontal plane [15].

In the next section, we summarize the strong-focussing IBS theory for bunched beams contained in Ref. [5] and Bane's high energy approximation to that theory, which we call *Bane*. In Section 3, we summarize the weak-focussing IBS theory for bunched beams contained in Ref. [2] and the Bane-inspired high energy approximation to that theory called *Completely Integrated Modified Piwinski (CIMP)* [11]. Finally, in Section 4, we propose a new high energy approximation for flat electron and positron beams and compare the results with the Bane and CIMP high energy approximations for the Advanced Test Facility (ATF) at Japan's KEK. The ATF is a prototype damping ring for the next generation of electron-positron linear colliders.

2. Strong-focussing IBS Theory and Bane's High Energy Approximation

For bunched beams in strong-focussing lattices, Ref. [5] gives the following growth rates (inverse time constants) for a beam's phase space:

$$\frac{1}{T_p} \equiv \frac{1}{\sigma_p}\frac{d\sigma_p}{dt} \quad (1)$$

$$\frac{1}{T_h} \equiv \frac{1}{\varepsilon_h^{\frac{1}{2}}}\frac{d\varepsilon_h^{\frac{1}{2}}}{dt} \quad (2)$$

$$\frac{1}{T_v} \equiv \frac{1}{\varepsilon_v^{\frac{1}{2}}}\frac{d\varepsilon_v^{\frac{1}{2}}}{dt} \quad (3)$$

with

$$\frac{1}{T_i} = 4\pi A(log)\,\langle \int_0^\infty d\lambda \frac{\lambda^{\frac{1}{2}}}{[det(L+\lambda I)]^{\frac{1}{2}}} \{TrL^iTr(\frac{1}{L+\lambda I}) - 3Tr[L^i(\frac{1}{L+\lambda I})]\}\,\rangle \quad (4)$$

where T_p, T_h, and T_v are the growth times for the relative energy spread and horizontal and vertical emittances (phase spaces), respectively, i represents p, h, or v, $\langle\cdots\rangle$ indicates that the integral is to be averaged around the accelerator lattice, and

$$A = \frac{r_0^2 cN}{64\pi^2\beta^3\gamma^4\varepsilon_h\varepsilon_v\sigma_s\sigma_p} \quad (5)$$

$$L = L^{(p)} + L^{(h)} + L^{(v)} \quad (6)$$

$$L^{(p)} = \frac{\gamma^2}{\sigma_p^2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (7)$$

$$L^{(h)} = \frac{\beta_h}{\varepsilon_h}\begin{pmatrix} 1 & -\gamma\phi_h & 0 \\ -\gamma\phi_h & \frac{\gamma^2\mathcal{H}_h}{\beta_h} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (8)$$

$$L^{(v)} = \frac{\beta_v}{\varepsilon_v}\begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{\gamma^2\mathcal{H}_v}{\beta_v} & -\gamma\phi_v \\ 0 & -\gamma\phi_v & 1 \end{pmatrix}. \quad (9)$$

(log) is the Coulomb logarithm which we define later, N is the number of particles in a bunch, r_0 is the classical radius of the charged particle, c is the speed of light in vacuum, β is the particle speed divided by c, γ is the particle energy divided by the rest mass, $\varepsilon_{h,v} \equiv \sigma_{h,v}^2/\beta_{h,v}$ are the transverse emittances, $\sigma_{h,v}$ are the rms transverse beam sizes, σ_s is the rms bunch length, and σ_p is the relative energy spread. Also, we have the horizontal dispersion invariant $\mathcal{H}_h = [\eta_h^2 + (\beta_h\eta_h' - \frac{1}{2}\beta_h'\eta_h)^2]/\beta_h$ and the function $\phi_h = \eta_h' - \frac{1}{2}\beta_h'\eta_h/\beta_h$, with similar expressions for the vertical functions. Finally, $\beta_{h,v}$ and $\eta_{h,v}$ are the betatron and dispersion accelerator lattice functions, respectively.

Bane [10] derived the following high energy approximations to the above formulas:

$$\frac{1}{T_p} \approx \frac{r_0^2 cN(log)}{16\gamma^3\varepsilon_h^{\frac{3}{4}}\varepsilon_v^{\frac{3}{4}}\sigma_s\sigma_p^3}\,\langle\,\sigma_H g_{Bane}(\frac{a}{b})(\beta_h\beta_v)^{-\frac{1}{4}}\,\rangle \quad (10)$$

$$\frac{1}{T_{h,v}} \approx \frac{\sigma_p^2\,\langle\,\mathcal{H}_{h,v}\rangle}{\varepsilon_{h,v}}\,\frac{1}{T_p} \quad (11)$$

where

$$\frac{1}{\sigma_H^2} = \frac{1}{\sigma_p^2} + \frac{\mathcal{H}_h}{\varepsilon_h} + \frac{\mathcal{H}_v}{\varepsilon_v} \quad (12)$$

$$a = \frac{\sigma_H}{\gamma}\sqrt{\frac{\beta_h}{\varepsilon_h}} \quad (13)$$

$$b = \frac{\sigma_H}{\gamma}\sqrt{\frac{\beta_v}{\varepsilon_v}} \quad (14)$$

with

$$g_{Bane}(\alpha) = \frac{2\sqrt{\alpha}}{\pi}\int_0^\infty \frac{du}{\sqrt{1+u^2}\sqrt{\alpha^2+u^2}} \ . \quad (15)$$

Eq. (11) is not useful for calculating T_v when the vertical dispersion vanishes.

We have discussed the strong-focussing IBS theory contained in Ref. [5] and Bane's high energy approximation to that theory [10]. In the next section, we summarize the weak-focussing IBS theory for bunched beams [2] and the Bane-inspired high energy approximation to that theory Ref. [11], which we call the *Completely Integrated Modified Piwinski (CIMP)* approximation.

3. Weak-focussing IBS Theory and the Completely Integrated Modified Piwinski (CIMP) Approximation

Piwinski's weak-focussing theory of intrabeam scattering is summarized nicely in Ref. [3]. The relative energy spread and transverse emittance growth rates are given by

$$\frac{1}{T_p} = A \left\langle \frac{\sigma_h^2}{\sigma_p^2} f(\tilde{a},\tilde{b},\tilde{q}) \right\rangle \quad (16)$$

$$\frac{1}{T_h} = A \left\langle f(\frac{1}{\tilde{a}},\frac{\tilde{b}}{\tilde{a}},\frac{\tilde{q}}{\tilde{a}}) + \frac{\eta_h^2\sigma_h^2}{\beta_h\varepsilon_h} f(\tilde{a},\tilde{b},\tilde{q}) \right\rangle \quad (17)$$

$$\frac{1}{T_v} = A \left\langle f(\frac{1}{\tilde{b}},\frac{\tilde{a}}{\tilde{b}},\frac{\tilde{q}}{\tilde{b}}) + \frac{\eta_v^2\sigma_h^2}{\beta_v\varepsilon_v} f(\tilde{a},\tilde{b},\tilde{q}) \right\rangle \quad (18)$$

where A is defined the same as in Eq. (5) and

$$\frac{1}{\sigma_h^2} = \frac{1}{\sigma_p^2} + \frac{\eta_h^2}{\beta_h\varepsilon_h} + \frac{\eta_v^2}{\beta_v\varepsilon_v} \quad (19)$$

$$\tilde{a} = \frac{\sigma_h}{\gamma}\sqrt{\frac{\beta_h}{\varepsilon_h}} \quad (20)$$

$$\tilde{b} = \frac{\sigma_h}{\gamma}\sqrt{\frac{\beta_v}{\varepsilon_v}} \quad (21)$$

$$\tilde{q} = \sigma_h\beta\sqrt{\frac{2d}{r_0}} \ . \quad (22)$$

The maximum impact parameter d is usually taken to be the vertical beam size and the Piwinski scattering function f is obtained from Ref. [2] as

$$f(\tilde{a},\tilde{b},\tilde{q}) = 2\int_0^\infty\int_0^\pi\int_0^{2\pi} e^{-r[\cos^2\theta+(\tilde{a}^2\cos^2\phi+\tilde{b}^2\sin^2\phi)\sin^2\theta]} \ln(\tilde{q}^2 r)(1-3\cos^2\theta)\sin\theta d\phi d\theta dr \quad (23)$$

where f satisfies the following relations:

$$f(\tilde{a},\tilde{b},\tilde{q}) = f(\tilde{b},\tilde{a},\tilde{q}) \quad (24)$$

$$f(\tilde{a},\tilde{b},\tilde{q}) + \frac{1}{\tilde{a}^2} f(\frac{1}{\tilde{a}},\frac{\tilde{b}}{\tilde{a}},\frac{\tilde{q}}{\tilde{a}}) + \frac{1}{\tilde{b}^2} f(\frac{1}{\tilde{b}},\frac{\tilde{a}}{\tilde{b}},\frac{\tilde{q}}{\tilde{b}}) = 0. \quad (25)$$

Evans and Zotter [4] performed two of the integrals in Piwinski's scattering function giving

$$f(\tilde{a},\tilde{b},\tilde{q}) = 8\pi\int_0^1 du \frac{(1-3u^2)}{PQ} \{2ln[\frac{\tilde{q}}{2}(\frac{1}{P}+\frac{1}{Q})] - 0.577\ldots\} \quad (26)$$

with

$$P^2 = \tilde{a}^2 + (1-\tilde{a}^2)u^2 \quad (27)$$

$$Q^2 = \tilde{b}^2 + (1-\tilde{b}^2)u^2. \quad (28)$$

To account for the lattice parameter variations around the accelerator in the high energy limit, Bane proposed the following replacements in the Piwinski theory [10]:

$$\frac{\eta_h^2}{\beta_h} \longrightarrow \mathcal{H}_h = [\eta_h^2 + (\beta_h\eta_h' - \frac{1}{2}\beta_h'\eta_h)^2]/\beta_h$$

which means $\sigma_h, \tilde{a}, \tilde{b}$ from this section become σ_H, a, b from the previous section. For example,

$$\frac{1}{\sigma_h^2} = \frac{1}{\sigma_p^2} + \frac{\eta_h^2}{\beta_h\varepsilon_h} + \frac{\eta_v^2}{\beta_v\varepsilon_v} \longrightarrow \frac{1}{\sigma_H^2} = \frac{1}{\sigma_p^2} + \frac{\mathcal{H}_h}{\varepsilon_h} + \frac{\mathcal{H}_v}{\varepsilon_v}. \quad (29)$$

Also,

$$\tilde{q} = \sigma_h \beta \sqrt{\frac{2d}{r_0}} \longrightarrow q = \sigma_H \beta \sqrt{\frac{2d}{r_0}} \, . \qquad (30)$$

In Ref. [11], it is shown that one can derive the following completely integrated expressions, called the *Completely Integrated Modified Piwinski (CIMP)* approximations, for the beam phase space growth rates in Bane's modified Piwinski theory:

$$\frac{1}{T_p} \approx 2\pi^{\frac{3}{2}} A(log) \, \langle \, \frac{\sigma_H^2}{\sigma_p^2} \, (\frac{g(\frac{b}{a})}{a} + \frac{g(\frac{a}{b})}{b}) \, \rangle \quad (31)$$

$$\frac{1}{T_h} \approx 2\pi^{\frac{3}{2}} A(log) \, \langle \, -ag(\frac{b}{a}) + + \frac{\mathcal{H}_h \sigma_H^2}{\varepsilon_h} \, (\frac{g(\frac{b}{a})}{a} + \frac{g(\frac{a}{b})}{b}) \, \rangle \quad (32)$$

$$\frac{1}{T_v} \approx 2\pi^{\frac{3}{2}} A(log) \, \langle \, -bg(\frac{a}{b}) + + \frac{\mathcal{H}_v \sigma_H^2}{\varepsilon_v} \, (\frac{g(\frac{b}{a})}{a} + \frac{g(\frac{a}{b})}{b}), \, \rangle, \quad (33)$$

where σ_H, a, b, and q, are defined in Eqs. (12-14), and (30). Also, all three integrals in the Piwinski scattering function f have been performed giving the scattering function g, which was derived in Ref. [6] and is given by

$$g(\omega) = \sqrt{\frac{\pi}{\omega}} \;\; [\, P_{-\frac{1}{2}}^{\;0}(\frac{\omega^2+1}{2\omega}) \pm \frac{3}{2} P_{-\frac{1}{2}}^{-1}(\frac{\omega^2+1}{2\omega}) \,], \quad (34)$$

where $P_\nu^{-\mu}$ are the Associated Legendre Functions. One takes the plus sign for $\omega \geq 1$ and the minus sign for $\omega \leq 1$. Note that $g(\omega) \to \sqrt{\pi}$ as $\omega \to 1$ from above or below. We have found that the Type 3 Associated Legendre Functions are the correct ones to use. In any event, it is important to check that one can reproduce Table I in Ref. [6] for the function g.

There is a Coulomb log factor that is common to all IBS theories. It appears in Eqs. (4) and (31)-(33) and is generally taken to be

$$(log) \equiv \ln(\frac{q^2}{a^2}) \approx \ln[\, \frac{\gamma^2 \sigma_v \varepsilon_h}{r_0 \beta_h} \,]. \qquad (35)$$

It has been shown [10, 11] that both Bane's high energy approximation to IBS and the CIMP approximation are in excellent agreement with the more complete strong-focussing theory [5] in the high energy limit, with the CIMP approximation being valid even for zero vertical dispersion and no coupling of the horizontal and vertical beam motion.

In the next section, we describe a new, even simpler high energy approximation to IBS for flat electron and positron beams.

4. A New High Energy IBS Approximation for Flat Electron and Positron Beams

To arrive at his high energy IBS formulae discussed in Section II, Bane makes two approximations to the strong-focussing theory contained in Eq. (4): (i) drops the second term in Eq. (4) and (ii) drops all off-diagonal elements in all matrices. While the first approximation can lead to errors as high as 100% in the vertical growth rate, we find that the second approximation is a good one for high energy, low emittance flat beams with

$$a^2 << b^2 << 1, \qquad (36)$$

which is typical of low-emittance electron and positron damping rings and synchrotron light sources, where a and b are defined in Eqs. (13) and (14). In fact, for the ATF at KEK, we average the parameters around the accelerator ring and get

$$< a^2 > \; = \; 0.00011 \qquad (37)$$

$$< b^2 > \; = \; 0.03192. \qquad (38)$$

We call dropping all off-diagonal elements in all matrices the *Diagonal Matrices (DM)* Approximation.

From Eq. (3.4) in Ref. [5], one can write the IBS emittance growth rates in terms of a diffusion matrix K_{ij} as follows:

$$\frac{1}{T_a} = \sum_{ij} K_{ij} L_{ji}^{(a)} \qquad (39)$$

where the matrices $L^{(a)}$ are defined in Eqs. (7-9) and

$$K_{ij} = 4\pi A(log) \, \langle \int_0^\infty d\lambda \frac{\lambda^{\frac{1}{2}}}{[det(L+\lambda I)]^{\frac{1}{2}}} \{\delta_{ij} Tr(\frac{1}{L+\lambda I}) - 3(\frac{1}{L+\lambda I})_{ij}\} \, \rangle \, . \quad (40)$$

After using the change of variable $\lambda = \frac{\gamma^2}{\sigma_H^2}\lambda'$ as suggested by Bane [10], employing the DM approximation, and performing the integrations, we can write the diffusion matrix elements as follows:

$$K_{ij} = 0 \text{ for } i \neq j \quad (41)$$

$$K_{11} = 4\pi A(log)\frac{\sigma_H^2}{\gamma^2}[(-2)M_1(a,b) + M_2(a,b) + +M_3(a,b)] \quad (42)$$

$$K_{22} = 4\pi A(log)\frac{\sigma_H^2}{\gamma^2}[M_1(a,b) + M_2(a,b) + +(-2)M_3(a,b)] \quad (43)$$

$$K_{33} = 4\pi A(log)\frac{\sigma_H^2}{\gamma^2}[M_1(a,b) + (-2)M_2(a,b) + +M_3(a,b)], \quad (44)$$

where

$$M_1(a,b) = \int_0^\infty d\lambda' \frac{\sqrt{\lambda'}}{\sqrt{(\lambda'+a^2)^3(\lambda'+b^2)(\lambda'+1)}} = \frac{2}{(b^2-a^2)}\sqrt{\frac{b^2}{(1-a^2)}}\;[F(\varphi,k) - E(\varphi,k)] \quad (45)$$

$$M_2(a,b) = \int_0^\infty d\lambda' \frac{\sqrt{\lambda'}}{\sqrt{(\lambda'+a^2)(\lambda'+b^2)^3(\lambda'+1)}} = \frac{2\sqrt{(1-a^2)b^2}}{(b^2-a^2)(1-b^2)}\;E(\varphi,k) - -\frac{2a^2}{(b^2-a^2)\sqrt{b^2(1-a^2)}}\;F(\varphi,k) - \frac{2}{(1-b^2)} \quad (46)$$

$$M_3(a,b) = \int_0^\infty d\lambda' \frac{\sqrt{\lambda'}}{\sqrt{(\lambda'+a^2)(\lambda'+b^2)(\lambda'+1)^3}} = \frac{2}{(b^2-1)}\sqrt{\frac{b^2}{(1-a^2)}}\;E(\varphi,k) + \frac{2}{(1-b^2)}, \quad (47)$$

and $F(\varphi,k)$ and $E(\varphi,k)$ are elliptic integrals of the first and second kinds, respectively. They are defined by

$$F(\varphi,k) \equiv \int_0^\varphi \frac{d\alpha}{\sqrt{1-k^2\sin^2\alpha}} \quad (48)$$

$$E(\varphi,k) \equiv \int_0^\varphi \sqrt{1-k^2\sin^2\alpha}, \quad (49)$$

where in our case

$$\varphi = \sin^{-1}\sqrt{1-a^2} \sim \sin^{-1}1 = \frac{\pi}{2} \quad (50)$$

$$k = \sqrt{\frac{(b^2-a^2)}{b^2(1-a^2)}} \quad (51)$$

The complete elliptic integrals are defined by

$$\tilde{K}(k) \equiv F(\frac{\pi}{2},k) \quad (52)$$

$$\tilde{E}(k) \equiv E(\frac{\pi}{2},k)\ . \quad (53)$$

In our case, $k \sim 1$. Although $\tilde{E}(1) = 1$, $\tilde{K}(1)$ diverges. Hence, we must use the approximation

$$\tilde{K}(\sqrt{\frac{(b^2-a^2)}{b^2(1-a^2)}}) \sim \tilde{K}(1-\frac{a^2}{2b^2})\ . \quad (54)$$

After a series of manipulations, the Diagonal Matrices approximation for the elements of the diffusion matrix for flat beams can be brought to the following simple form:

$$K_{ij} = 0 \text{ for } i \neq j \quad (55)$$

$$K_{11} = 4\pi A(log)\frac{\sigma_H^2}{\gamma^2}[\frac{6}{b} - 2b - \frac{4}{b}\tilde{K}(1-\frac{a^2}{2b^2})] \quad (56)$$

$$K_{22} = 4\pi A(log)\frac{\sigma_H^2}{\gamma^2}[-6 + 4b + \frac{2}{b}\tilde{K}(1-\frac{a^2}{2b^2})] \quad (57)$$

$$K_{33} = 4\pi A(log)\frac{\sigma_H^2}{\gamma^2}[6 - 2b - \frac{6}{b} + \frac{2}{b}\tilde{K}(1-\frac{a^2}{2b^2})]. \quad (58)$$

In Eqs. (56)-(58), we can use Hastings' approximation for $\tilde{K}(k)$ [16]:

$$\tilde{K}(k) = \sum_{n=0}^{4} a_n t^n - \ln t \sum_{n=0}^{4} b_n t^n + \varepsilon(k)\ , \quad (59)$$

where $t = 1 - k^2$, $|\varepsilon(k)| \leq 2 \times 10^{-8}$, and

$$a_0 = 1.38629436112 \quad b_0 = 0.5 \quad (60)$$

$$a_1 = 0.09666344259 \quad b_1 = 0.12498593597 \quad (61)$$

$$a_2 = 0.03590092383 \quad b_2 = 0.06880248576 \quad (62)$$

$$a_3 = 0.03742563713 \quad b_3 = 0.03328355346 \quad (63)$$

$$a_4 = 0.01451196212 \quad b_4 = 0.00441787012 \quad (64)$$

Using Eq. (39), Eqs. (55)-(64), and the methods discussed in Ref. [11], we have calculated the IBS growth rates and compared them to the more complete theory [5] and to the Bane and CIMP high energy approximations for the Advanced Test Facility (ATF) at Japan's KEK. The nominal beam parameters for the ATF are contained in Table 1.

Table 1. Nominal beam parameters for the ATF damping ring [17].

Beam energy	1.28 GeV
Electrons per bunch	10^{10}
Horizontal emittance	1.15 nm
Vertical emittance	4.03 pm
Relative energy spread	5.47×10^{-4}
Bunch length (rms)	5.59 mm

Averaging the IBS emittance growth rates around the ATF lattice, we obtain the rates contained in Table 2.

Table 2. Comparison of the IBS emittance growth rates for the more complete theory of Ref. [5] and the Bane, CIMP, and Diagonal Matrices (DM) high energy approximations. Quantities are in $\sec^{-1}$.

Approximation	$\frac{1}{T_p}$	$\frac{1}{T_h}$	$\frac{1}{T_v}$
Ref. [5]	352.218	261.949	123.011
Bane	400.862	285.655	130.987
CIMP	399.676	287.066	137.236
DM	363.484	267.618	127.647

We see that the DM approximation gives better agreement with the more complete theory of Ref. [5] than the other approximations. We also note that it involves the use of elliptic integrals as suggested by Ref. [5].

5. Conclusion

We have derived a new high energy approximation for the emittance growth rates for electron and positron beams with vertical emittance much less than the horizontal, which is typical for modern electron-positron accelerators, such as synchrotron light sources and damping rings for colliders. Such approximations are extremely useful in that they greatly reduce the time it takes computers to average growth rates around accelerator lattices. Often hours of computing time are reduced to minutes, or even seconds. The new Diagonal Matrices approximation is easy to use and appears to offer an improvement over the Bane and CIMP high energy approximations.

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